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ROMAN DOMINATION ON SOME GRAPHS

Derya Doğan Durgun[1*], **Emre Niyazi Toprakkaya**[2]

[1]Arts and Science Faculty, Manisa Celal Bayar University, Manisa, Türkiye
[2]Institute of Natural and Applied Sciences, Manisa Celal Bayar University, Manisa, Türkiye

Abstract. Let $G = (V, E)$ be a graph. A Roman dominating function (RDF) $f : V \to \{0, 1, 2\}$ in satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is $f(V) = \sum_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. In this paper, the Roman domination numbers of some graphs are given.

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[*]**Corresponding author:** Derya, Doğan Durgun, Arts and Science Faculty, Manisa Celal Bayar University, Manisa, Türkiye, e-mail: *derya.dogan@cbu.edu.tr*
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1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$ (shortly V and E respectively). The order of G is $|V| = n$. The open neighborhood of a vertex is $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and closed neighborhood of v is $N_G[v] = N[v] = N(v) \cup \{v\}$ (Henning, 2003). The degree of a vertex v is $deg_G(v) = deg(v) = |N(v)|$. The denotation of the minimum and maximum degree of a graph G is by $\delta(G)$ and $\Delta(G)$, respectively. For a set $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A subset S of vertices of G is a dominating set if $N[S] = V$. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set of G. Such a set of G is called a $\gamma(G) - set$ (Dogan Durgun & Lokcu, 2020). Many varieties of dominating sets had studied in the book "Fundamentals of Domination in Graphs" (Haynes et al., 1998).

We consider the Roman domination number, defined by Stewart (1999). Roman domination appears to be a variety of both historical and mathematical interests (ReVelle & Rosing, 2000).

On a graph G, while $f : V \to \{0, 1, 2\}$, if every vertex u for which $f(u) = 0$, is adjacent to at least one vertex v for which $f(v) = 2$, then we call f a Roman dominating function (RDF) (Cockayne et al., 2004; Dreyer, 2000). The weight of an RDF is the value $w(f) = \Sigma_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. $\gamma_R(G) - function$ is a Roman dominating function of G with weight $\gamma_R(G)$ Henning & Hedetniemi (2003). A Roman dominating function $f : V \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V | f(v) = i\}$. Its weight is $w(f) = |V_1| + 2|V_2|$ (Chambers et al., 2009).

To get a better understanding of how Roman domination works in graphs, consider some of the well-known types, the path, and the cycle graphs. The results had already been given for $\gamma_R(C_n) = \gamma_R(P_n) = \lceil \frac{2n}{3} \rceil$ Cockayne et al. (2004). For example, take C_5 into consideration.

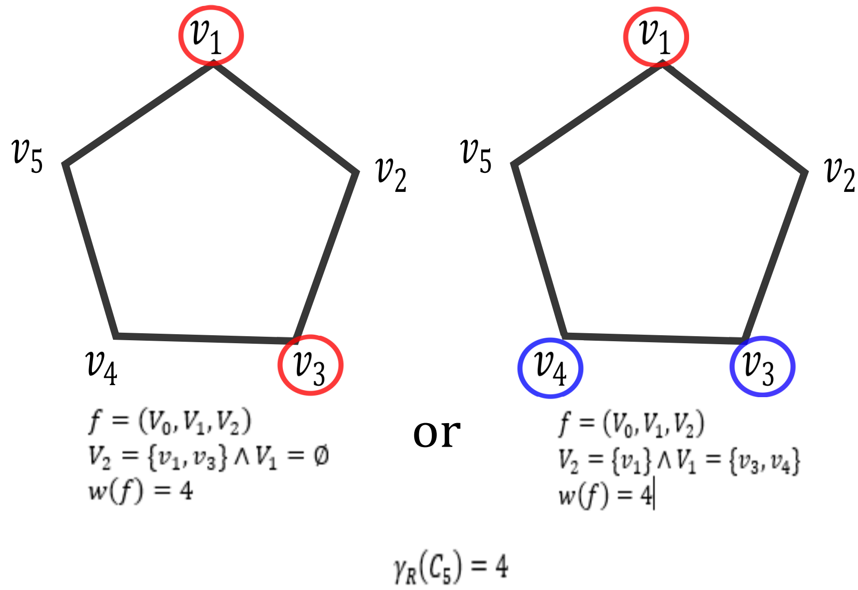

Figure 1: An example for C_5

To minimize the weight of an RDF on C_5, v_1 should be taken into V_2, so it dominates itself, and also v_2 and v_5. One of the rest vertices of C_5 should be taken into V_2 to dominate itself and its neighbor. Or they both should be taken into V_1 to dominate themselves. In both circumstances, $\gamma_R(C_5) = 4$ as expected.

In the meantime, many interesting, and broadening works have been studied on graph theoretic parameters in tree-structured graphs (Aytaç & Turacı, 2021; Çiftçi & Aytaç, 2018; Dogan Durgun & Lokcu, 2020).

Roman domination is a very important domination issue in graphs. It has many uses in real-life networks. Take the COVID-19 pandemic, a highly prior subject on the agenda, into consideration as an example of Roman domination. Firstly, let every healthy individual be a member of the V_1 set before the pandemic begins. Just as the healthcare teams try to do in the filiation (contact tracing) studies, for the first positive case that occurs, we should take; the individual with the disease into V_2 set and those who are determined to be in contact with this person into V_0 set. When we model such a graph, monitoring the course of the disease, controlling the adequacy of hospital capacities, and protecting healthy individuals by separating them from those with the disease, become much more possible. The calculated Roman domination number for this model can be one data that gives us information about the limit values of herd immunity. Also, we can use a graph model like this in the decision-making process about where establishing pandemic hospitals should be. By analyzing the structure and properties of the graph, including connectivity and domination, we can identify areas with a higher concentration of infected individuals and allocate resources accordingly. For more information about the domination parameters, and terminologies here (Haynes et al., 1998; West, 2001).

In this paper, the Roman domination number of some graphs is given with their proofs.

2 Roman Domination Numbers of Some Graphs

In this section, the Roman domination numbers of comet, double comet and comb graphs are given.

Definition 1. ***(Comet Graph):*** *For $t \geq 2$ and $r \geq 1$, the comet graph $C_{t,r}$ with $t + r$ vertices is the graph obtained by identifying one end of the path P_t with the center of the star $K_{1,r}$, and Figure 2 shows $C_{t,r}$ (Bagga et al., 1992).*

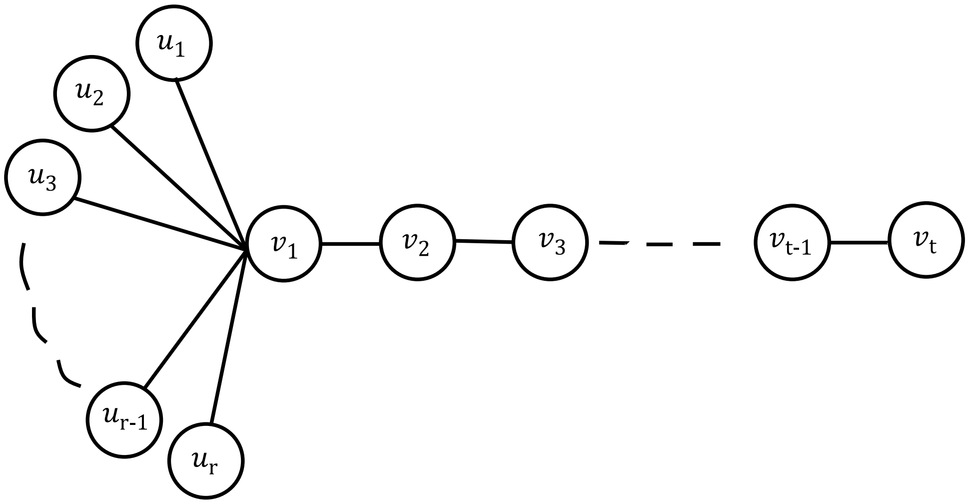

Figure 2: Comet Graph $C_{t,r}$

Theorem 1. *Let $G = C_{t,r}$ be a comet graph where $t \geq 2$ and $r \geq 1$. Then the Roman domination number of G is equal to;*

$$\gamma_R(C_{t,r}) = \begin{cases} 2\frac{t}{3}+1 & t\equiv 0 (mod\ 3) \\ 2\lceil \frac{t}{3} \rceil & otherwise \end{cases}$$

Proof. Roman domination number of the comet graph is considered in three cases. For all cases, assume that $f = (V_0, V_1, V_2)$ is a $\gamma_R - function$ of G.

(1) $t\equiv 0 \pmod 3$.

In order to dominate $u_1, u_2, u_3, \ldots, u_r$, v_1 and v_2 vertices, v_1 vertex should be taken into V_2. To dominate v_{t-2}, v_{t-1} and v_{t-3} vertices, v_{t-2} vertex should be taken into V_2. To dominate v_t vertex, v_t itself should be taken into V_1. For the rest vertices of the graph which are not dominated, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. Then $f = (V_0, V_1 = \{v_t\}, V_2 = \{v_1, v_4, v_7, \ldots, v_{t-2}\})$.

So that $f(V) = 1 + 2(\frac{t-2-1}{3} + 1)$ then we get $\gamma_R(C_{t,r}) \leq 2\frac{t}{3} + 1$.

On the other hand, let f not be a $\gamma_R - function$ and by deleting v_t vertex from $V_1 = \{v \in V : f(v) = 1\}$, let $V_1 = \emptyset$. Since $f_1(v) \neq 2$ for $\forall v \in N(v_t)$, obtained function $f_1 = (V_0 \cup \{v_t\}, \emptyset, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(C_{t,r}) \geq 2\frac{t}{3} + 1$. For f_1 function to be an RDF, v_t vertex should be taken into V_2; $f_1 = (V_0, \emptyset, V_2 \cup \{v_t\})$. Hence we get $f_1(V) = 2\frac{t}{3} + 2$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(C_{t,r}) \geq 2\frac{t}{3} + 1$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_4 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(v_4)$, obtained function $f_2 = (V_0 \cup \{v_4\}, V_1, V_2 - \{v_4\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(C_{t,r}) \geq 2\frac{t}{3}+1$. For f_2 function to be an RDF, v_3, v_4, v_5 vertices should be taken into V_1; $f_2 = (V_0 - \{v_3, v_5\}, V_1 \cup \{v_3, v_4, v_5\}, V_2 - \{v_4\})$. Hence we get $f_2(V) = 2\frac{t}{3} + 2$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(C_{t,r}) \geq 2\frac{t}{3} + 1$.

Consequently $\gamma_R(C_{t,r}) = 2\frac{t}{3} + 1$.

(2) $t\equiv 1 \pmod 3$.

i) In order to dominate $u_1, u_2, u_3, \ldots, u_r$, v_1 and v_2 vertices, v_1 vertex should be taken into V_2. To dominate v_{t-1}, v_{t-2} and v_t vertices, v_{t-1} vertex should be taken into V_2. For the rest vertices of the graph which are not dominated, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. Because of the v_{t-2} vertex is dominated by v_{t-3} vertex at the same time, taking v_t vertex into V_2 instead of v_{t-1} vertex does not change the result. Then $f = (V_0, V_1 = \emptyset, V_2 = \{v_1, v_4, v_7, \ldots, v_{t-1}\})$, or $f = (V_0, V_1 = \emptyset, V_2 = \{v_1, v_4, v_7, \ldots, v_t\})$.

So that $f(V) = 2(\frac{t-3-1}{3} + 1 + 1)$ then we get $\gamma_R(C_{t,r}) \leq 2\lceil \frac{t}{3} \rceil$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_4 vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_4)$, obtained function $f_1 = (V_0 \cup \{v_4\}, \emptyset, V_2 - \{v_4\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(C_{t,r}) \geq 2\lceil \frac{t}{3} \rceil$. For f_1 function to be an RDF, v_3, v_4, v_5 vertices should be taken into V_1; $f_1 = (V_0 - \{v_3, v_5\}, V_1 \cup \{v_3, v_4, v_5\}, V_2 - \{v_4\})$. Hence we get $f_1(V) = 2\lceil \frac{t}{3} \rceil + 1$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$.

In this case $\gamma_R(C_{t,r}) \geq 2\lceil \frac{t}{3} \rceil$.

ii) In order to dominate $u_1, u_2, u_3, \ldots, u_r$, v_1 and v_2 vertices, v_1 vertex should be taken into V_2. To dominate v_{t-3}, v_{t-4} and v_{t-2} vertices, v_{t-3} vertex should be taken into V_2. To dominate v_t and v_{t-1} vertices, v_t and v_{t-1} themselves should be taken into V_1. For the rest vertices of the graph which are not dominated, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. Then $V_2 = \{v_1, v_4, v_7, \ldots, v_{t-3}\}$ and $V_1 = \{v_{t-1}, v_t\}$.

Therefore $f(V) = 2(\frac{t-3-1}{3} + 1) + 2$ then we get $\gamma_R(C_{t,r}) \leq 2\lceil \frac{t}{3} \rceil$.

Let f not be a $\gamma_R - function$ and delete any vertex from V_2, then the result will be the same as above. So that any vertex of V_1, such as v_t vertex, should be deleted. Since $f_2(v) \neq 2$ for $\forall v \in N(v_t)$, obtained function $f_2 = (V_0 \cup \{v_t\}, V_1 - \{v_t\}, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(C_{t,r}) \geq 2\lceil \frac{t}{3} \rceil$. For f_2 function to be an RDF, v_t vertex should be taken into V_2; $f_2 = (V_0, V_1 - \{v_t\}, V_2 \cup \{v_t\})$. Hence we get $f_2(V) = 2\lceil \frac{t}{3} \rceil + 1$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(C_{t,r}) \geq 2\lceil \frac{t}{3} \rceil$.

Consequently $\gamma_R(C_{t,r}) = 2\lceil \frac{t}{3} \rceil$.

(3) $t \equiv 2 \pmod 3$.

In order to dominate $u_1, u_2, u_3, \ldots, u_r$, v_1 and v_2 vertices, v_1 vertex should be taken into V_2. To dominate v_{t-1}, v_{t-2} and v_t vertices, v_{t-1} vertex should be taken into V_2. For the rest vertices of the graph which are not dominated, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. Then $f = (V_0, V_1 = \emptyset, V_2 = \{v_1, v_4, v_7, \ldots, v_{t-1}\})$.

So that $f(V) = 2(\frac{t-1-1}{3} + 1)$ then we get $\gamma_R(C_{t,r}) \leq 2\lceil \frac{t}{3} \rceil$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_4 vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_4)$, obtained function $f_1 = (V_0 \cup \{v_4\}, \emptyset, V_2 - \{v_4\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(C_{t,r}) \geq 2\lceil \frac{t}{3} \rceil$. For f_1 function to be an RDF, v_3, v_4, v_5 vertices should be taken into V_1; $f_1 = (V_0 - \{v_3, v_5\}, V_1 \cup \{v_3, v_4, v_5\}, V_2 - \{v_4\})$. Hence we get $f_1(V) = 2\lceil \frac{t}{3} \rceil + 1$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$.

In this case $\gamma_R(C_{t,r}) \geq 2\lceil \frac{t}{3} \rceil$. Consequently $\gamma_R(C_{t,r}) = 2\lceil \frac{t}{3} \rceil$.

In the end, we have obtained;

$$\gamma_R(C_{t,r}) = \begin{cases} 2\frac{t}{3} + 1 & t \equiv 0 (\text{mod } 3) \\ 2\lceil \frac{t}{3} \rceil & otherwise \end{cases}$$

□

Definition 2. ***(Double Comet Graph)*** *A vertex of a graph is said to be pendant if its neighborhood contains exactly one vertex. An edge of a graph is said to be a pendant if one of its vertices is a pendant vertex. For $a, b \geq 1$, $n \geq a + b + 2$ by $DC(n, a, b)$ we denote a double comet graph, which is a tree composed of a path containing $n - a - b$ vertices with a pendant vertices attached to one of the ends of the path and b pendant vertices attached to the other end of the path, and Figure 3 shows $DC(n, a, b)$ (Cygan et al., 2011).*

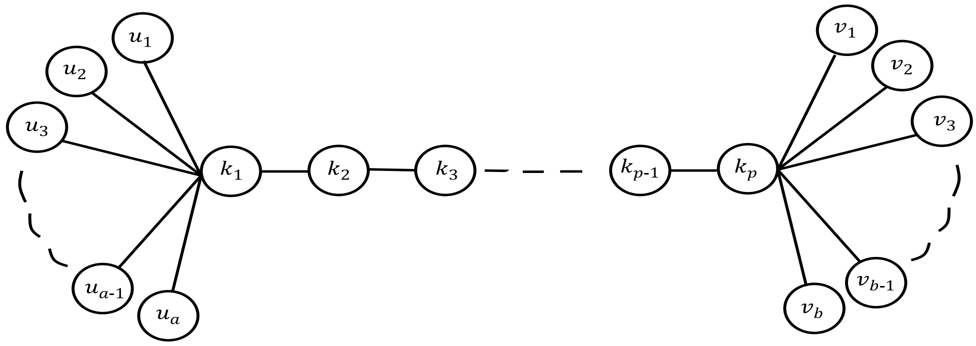

Figure 3: Double Comet Graph $DC(n, a, b)$

Theorem 2. *For $p = n - a - b$ and $p \neq 2$, let $G = DC(n, a, b)$ be a double comet graph. The Roman domination number of G is equal to;*

$$\gamma_R(DC(n,a,b)) = \begin{cases} 2(\frac{p}{3}+1) & p\equiv 0 \,(mod\ 3) \\ 2\lceil\frac{p}{3}\rceil & p\equiv 1 \,(mod\ 3) \\ 2\lceil\frac{p}{3}\rceil + 1 & p\equiv 2 \,(mod\ 3) \end{cases}$$

Proof. Let $G = \{u_1, \ldots, u_a, k_1, \ldots, k_p, v_1, \ldots, v_b\}$ be a graph as shown in Figure 3.

(1) $p\equiv 0 \pmod 3$.

In order to dominate $u_1, u_2, \ldots, u_a$ and k_1, k_2 vertices, k_1 vertex should be taken into V_2 and similarly to dominate $v_1, v_2, \ldots, v_b$ and k_p, k_{p-1} vertices, k_p vertex should be taken into V_2. For the rest vertices of the graph which are not dominated k_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$ Then $f = (V_0, V_1 = \emptyset, V_2 = \{k_1, k_4, \ldots, k_{p-2}, k_p\})$.

So that $f(V) = 2(\frac{p-2-1}{3} + 1 + 1)$ then we get $\gamma_R(DC(n, a, b)) \leq 2(\frac{p}{3} + 1)$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as k_4 vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(k_4)$, obtained function $f_1 = (V_0 \cup \{k_4\}, \emptyset, V_2 - \{k_4\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(DC(n, a, b)) \geq 2(\frac{p}{3} + 1)$. For f_1 function to be an RDF, k_3, k_4, k_5 vertices should be taken into V_1; $f_1 = (V_0 - \{k_3, k_5\}, V_1 \cup \{k_3, k_4, k_5\}, V_2 - \{k_4\})$. Hence we get $f_1(V) = 2(\frac{p}{3} + 1) + 1$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(DC(n, a, b)) \geq 2(\frac{p}{3} + 1)$.

Consequently $\gamma_R(DC(n, a, b)) = 2(\frac{p}{3} + 1)$.

(2) $p\equiv 1 \pmod 3$.

In order to dominate $u_1, u_2, \ldots, u_a$ and k_1, k_2 vertices, k_1 vertex should be taken into V_2 and similarly to dominate $v_1, v_2, \ldots, v_b$ and k_p, k_{p-1} vertices, k_p vertex should be taken into V_2. For the rest vertices of the graph which are not dominated k_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$ Then $f = (V_0, V_1 = \emptyset, V_2 = \{k_1, k_4, \ldots, k_{p-3}, k_p\})$.

So that $f(V) = 2(\frac{p-3-1}{3} + 1 + 1)$ then we get $\gamma_R(DC(n, a, b)) \leq 2\lceil\frac{p}{3}\rceil$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as k_4 vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(k_4)$, obtained function $f_1 = (V_0 \cup \{k_4\}, \emptyset, V_2 - \{k_4\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(DC(n, a, b)) \geq 2\lceil\frac{p}{3}\rceil$. For f_1 function to be an RDF, k_3, k_4, k_5 vertices should be taken into V_1; $f_1 = (V_0 - \{k_3, k_5\}, V_1 \cup \{k_3, k_4, k_5\}, V_2 - \{k_4\})$. Hence we get $f_1(V) = 2\lceil\frac{p}{3}\rceil + 1$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(DC(n, a, b)) \geq 2\lceil\frac{p}{3}\rceil$.

Consequently $\gamma_R(DC(n, a, b)) = 2\lceil\frac{p}{3}\rceil$.

(3) $p\equiv 2 \pmod 3$.

In order to dominate $u_1, u_2, \ldots, u_a$ and k_1, k_2 vertices, k_1 vertex should be taken into V_2 and similarly to dominate $v_1, v_2, \ldots, v_b$ and k_p, k_{p-1} vertices, k_p vertex should be taken into V_2. To dominate k_{p-2} vertex, k_{p-2} vertex itself should be taken into V_1. For the rest vertices of the graph which are not dominated k_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$ Then $f = (V_0, V_1 = \{k_{p-2}\}, V_2 = \{k_1, k_4, \ldots, k_{p-4}, k_p\})$.

So that $f(V) = 2(\frac{p-4-1}{3} + 1 + 1) + 1$ then we get $\gamma_R(DC(n,a,b)) \leq 2\lceil \frac{p}{3} \rceil + 1$.

On the other hand, let f not be a $\gamma_R - function$ and by deleting k_{p-2} vertex from $V_1 = \{v \in V : f(v) = 1\}$, let $V_1 = \emptyset$. Since $f_1(v) \neq 2$ for $\forall v \in N(k_{p-2})$, obtained function $f_1 = (V_0 \cup \{k_{p-2}\}, \emptyset, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(DC(n,a,b)) \geq 2\lceil \frac{p}{3} \rceil + 1$. For f_1 function to be an RDF, k_{p-2} vertex should be taken into V_2; $f_1 = (V_0, \emptyset, V_2 \cup \{k_{p-2}\})$. Hence we get $f_1(V) = 2\lceil \frac{p}{3} \rceil + 2$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(DC(n,a,b)) \geq 2\lceil \frac{p}{3} \rceil + 1$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as k_4 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(k_4)$, obtained function $f_2 = (V_0 \cup \{k_4\}, V_1, V_2 - \{k_4\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(DC(n,a,b)) \geq 2\lceil \frac{p}{3} \rceil + 1$. For f_2 function to be an RDF, k_3, k_4, k_5 vertices should be taken into V_1; $f_2 = (V_0 - \{k_3, k_5\}, V_1 \cup \{k_3, k_4, k_5\}, V_2 - \{k_4\})$. Hence we get $f_2(V) = 2\lceil \frac{p}{3} \rceil + 2$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(DC(n,a,b)) \geq 2\lceil \frac{p}{3} \rceil + 1$. Consequently $\gamma_R(DC(n,a,b)) = 2\lceil \frac{p}{3} \rceil + 1$.

Through these three cases, we have obtained;

$$\gamma_R(DC(n,a,b)) = \begin{cases} 2(\frac{p}{3} + 1) & p \equiv 0 (\text{mod } 3) \\ 2\lceil \frac{p}{3} \rceil & p \equiv 1 (\text{mod } 3) \\ 2\lceil \frac{p}{3} \rceil + 1 & p \equiv 2 (\text{mod } 3) \end{cases}$$

The proof is also could be done using path structure after adding v_1 to V_2 set. There remains P_{t-2} graph, and we already know that $\gamma_R(P_n) = \lceil \frac{2n}{3} \rceil$. □

Definition 3. ***(Comb Graph):*** *The comb graph is the graph obtained from a path P_n by attaching pendant edge at each vertex of the path, and is denoted by P_n^+ (Gayathri et al., 2007).*

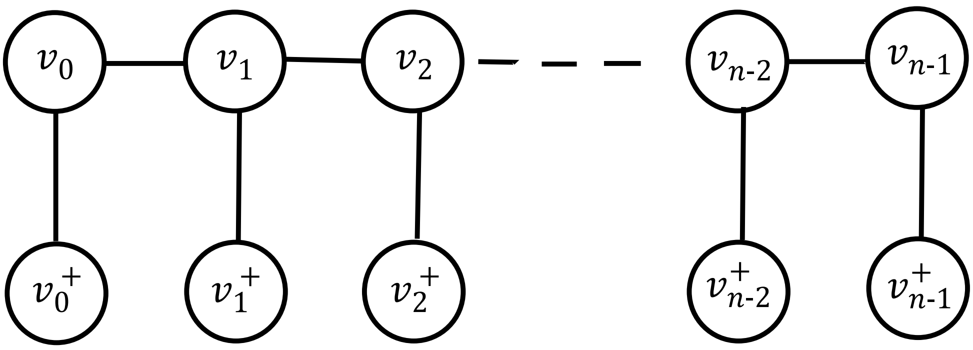

Figure 4: Comb Graph P_n^+

Theorem 3. *Let $G = P_n^+$ be a comb graph. The Roman domination number of G is equal to;*

$$\gamma_R(P_n^+) = \begin{cases} 4\frac{n}{3} & n \equiv 0 (\text{mod } 3) \\ 4\lfloor \frac{n}{3} \rfloor + 2 & n \equiv 1 (\text{mod } 3) \\ 4\lceil \frac{n}{3} \rceil - 1 & n \equiv 2 (\text{mod } 3) \end{cases}$$

Proof. The proof of the Roman domination number for comb graph, we use v_t^+ for the pendant vertices of v_t. Let G be a graph as shown in Figure 4. Assume that $f = (V_0, V_1, V_2)$ is a $\gamma_R - function$ of G.

(1) $n \equiv 0 \pmod 3$.

In order to dominate v_{t-1}, v_t, v_{t+1} and v_t^+ vertices, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. For the rest vertices of the graph which are not dominated, v_{t-1}^+

and v_{t+1}^+ vertices should be taken into V_1 which satisfy $t \equiv 1 (\text{mod } 3)$. Then $f = (V_0, V_1 = \{v_0^+, v_2^+, v_3^+, v_5^+, \ldots, v_{n-3}^+, v_{n-1}^+\}, V_2 = \{v_1, v_4, v_7, \ldots, v_{n-5}, v_{n-2}\})$.

So that $f(V) = (\frac{n-3-0}{3} + 1 + \frac{n-1-2}{3} + 1) + 2(\frac{n-2-1}{3} + 1)$ then we get $\gamma_R(P_n^+) \leq 4\frac{n}{3}$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_1 = \{v \in V : f(v) = 1\}$, such as v_0^+ vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_0^+)$, obtained function $f_1 = (V_0 \cup \{v_0^+\}, V_1 - \{v_0^+\}, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\frac{n}{3}$. For f_1 function to be an RDF, v_0^+ vertex should be taken into V_2; $f_1 = (V_0, V_1 - \{v_0^+\}, V_2 \cup \{v_0^+\})$. Hence we get $f_1(V) = 4\frac{n}{3} + 1$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\frac{n}{3}$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_1 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(v_1)$, obtained function $f_2 = (V_0 \cup \{v_1\}, V_1, V_2 - \{v_1\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\frac{n}{3}$. For f_2 function to be an RDF, v_0, v_1, v_2 and v_1^+ vertices should be taken into V_1; $f_2 = (V_0 - \{v_0, v_2, v_1^+\}, V_1 \cup \{v_0, v_1, v_2, v_1^+\}, V_2 - \{v_1\})$. Hence we get $f_2(V) = 4\frac{n}{3} + 2$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\frac{n}{3}$.

Consequently $\gamma_R(P_n^+) = 4\frac{n}{3}$.

(2) $n \equiv 1 \pmod 3$.

i) In order to dominate v_{t-1}, v_t, v_{t+1} and v_t^+ vertices, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. To dominate v_{n-1} and v_{n-1}^+ vertices, v_{n-1} or v_{n-1}^+ vertex should be taken into V_2. For the rest vertices of the graph which are not dominated, v_{t-1}^+ and v_{t+1}^+ vertices should be taken into V_1 which satisfy $t \equiv 1 (\text{mod } 3)$. Then $f = (V_0, V_1 = \{v_0^+, v_2^+, v_3^+, v_5^+, \ldots, v_{n-4}^+, v_{n-2}^+\},$
$V_2 = \{v_1, v_4, v_7, \ldots, v_{n-6}, v_{n-3}, v_{n-1}\})$ or $f = (V_0, V_1 = \{v_0^+, v_2^+, v_3^+, v_5^+, \ldots, v_{n-4}^+, v_{n-2}^+\}, V_2 = \{v_1, v_4, v_7, \ldots, v_{n-6}, v_{n-3}, v_{n-1}^+\})$.

Thus, $f(V) = (\frac{n-4-0}{3} + 1 + \frac{n-2-2}{3} + 1) + 2(\frac{n-3-1}{3} + 1 + 1)$ then we get $\gamma_R(P_n^+) \leq 4\lfloor \frac{n}{3} \rfloor + 2$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_1 = \{v \in V : f(v) = 1\}$, such as v_0^+ vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_0^+)$, obtained function $f_1 = (V_0 \cup \{v_0^+\}, V_1 - \{v_0^+\}, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$. For f_1 function to be an RDF, v_0^+ vertex should be taken into V_2; $f_1 = (V_0, V_1 - \{v_0^+\}, V_2 \cup \{v_0^+\})$. Hence we get $f_1(V) = 4\lfloor \frac{n}{3} \rfloor + 3$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_1 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(v_1)$, obtained function $f_2 = (V_0 \cup \{v_1\}, V_1, V_2 - \{v_1\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$. For f_2 function to be an RDF, v_0, v_1, v_2 and v_1^+ vertices should be taken into V_1; $f_2 = (V_0 - \{v_0, v_2, v_1^+\}, V_1 \cup \{v_0, v_1, v_2, v_1^+\}, V_2 - \{v_1\})$. Hence we get $f_2(V) = 4\lfloor \frac{n}{3} \rfloor + 4$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$.

ii) In order to dominate v_{t-1}, v_t, v_{t+1} and v_t^+ vertices, v_t vertices should be taken into V_2 which satisfy $t \equiv 1 (\text{mod } 3)$. To dominate v_{n-1} and v_{n-1}^+ vertices, v_{n-1} and v_{n-1}^+ vertices should be taken into V_1. For the rest vertices of the graph which are not dominated, v_{t-1}^+ and v_{t+1}^+ vertices should be taken into V_1 which satisfy $t \equiv 1 (\text{mod } 3)$. Then
$V_2 = \{v_1, v_4, v_7, \ldots, v_{n-6}, v_{n-3}\}$ and $V_1 = \{v_0^+, v_2^+, v_3^+, v_5^+, \ldots, v_{n-4}^+, v_{n-2}^+, v_{n-1}^+, v_{n-1}\}$.

So that $f(V) = (\frac{n-4-0}{3} + 1 + \frac{n-2-2}{3} + 1 + 2) + 2(\frac{n-3-1}{3} + 1)$ then we get $\gamma_R(P_n^+) \leq 4\lfloor \frac{n}{3} \rfloor + 2$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_1 = \{v \in V : f(v) = 1\}$, such as v_0^+ vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_0^+)$, obtained function $f_1 = (V_0 \cup \{v_0^+\}, V_1 - \{v_0^+\}, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$. For f_1 function to be an RDF, v_0^+ vertex should be taken into V_2; $f_1 = (V_0, V_1 - \{v_0^+\}, V_2 \cup \{v_0^+\})$. Hence we get $f_1(V) = 4\lfloor \frac{n}{3} \rfloor + 3$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_1 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(v_1)$, obtained function $f_2 = (V_0 \cup \{v_1\}, V_1, V_2 - \{v_1\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$. For

f_2 function to be an RDF, v_0, v_1, v_2 and v_1^+ vertices should be taken into V_1; $f_2 = (V_0 - \{v_0, v_2, v_1^+\}, V_1 \cup \{v_0, v_1, v_2, v_1^+\}, V_2 - \{v_1\})$. Hence we get $f_2(V) = 4\lfloor \frac{n}{3} \rfloor + 4$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lfloor \frac{n}{3} \rfloor + 2$.

Consequently, we could say that $\gamma_R(P_n^+) = 4\lfloor \frac{n}{3} \rfloor + 2$.

(3) $n \equiv 2 \pmod 3$.

i) In order to dominate v_{t-1}, v_t, v_{t+1} and v_t^+ vertices, v_t vertices should be taken into V_2 which satisfy $t \equiv 1(\text{mod } 3)$. For the rest vertices of the graph which are not dominated, v_{t-1}^+ and v_{t+1}^+ vertices should be taken into V_1 which satisfy $t \equiv 1(\text{mod } 3)$. Then $f = (V_0, V_1 = \{v_0^+, v_2^+, v_3^+, v_5^+, \ldots, v_{n-3}^+, v_{n-2}^+\}, V_2 = \{v_1, v_4, v_7, \ldots, v_{n-4}, v_{n-1}\})$.

So that $f(V) = (\frac{n-2-0}{3} + 1 + \frac{n-3-2}{3} + 1) + 2(\frac{n-1-1}{3} + 1)$ then we get $\gamma_R(P_n^+) \leq 4\lceil \frac{n}{3} \rceil - 1$.

On the other hand, let f not be a $\gamma_R - function$ and delete any vertex from $V_1 = \{v \in V : f(v) = 1\}$, such as v_0^+ vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_0^+)$, obtained function $f_1 = (V_0 \cup \{v_0^+\}, V_1 - \{v_0^+\}, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$. For f_1 function to be an RDF, v_0^+ vertex should be taken into V_2; $f_1 = (V_0, V_1 - \{v_0^+\}, V_2 \cup \{v_0^+\})$. Hence we get $f_1(V) = 4\lceil \frac{n}{3} \rceil$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_1 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(v_1)$, obtained function $f_2 = (V_0 \cup \{v_1\}, V_1, V_2 - \{v_1\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$. For f_2 function to be an RDF, v_0, v_1, v_2 and v_1^+ vertices should be taken into V_1; $f_2 = (V_0 - \{v_0, v_2, v_1^+\}, V_1 \cup \{v_0, v_1, v_2, v_1^+\}, V_2 - \{v_1\})$. Hence we get $f_2(V) = 4\lceil \frac{n}{3} \rceil + 1$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$.

ii) To dominate v_{n-2}, v_{n-1} and v_{n-2}^+ vertices, v_{n-2} vertex should be taken into V_2. To dominate v_{n-1}^+ vertex, v_{n-1}^+ vertex itself should be taken into V_1. For the rest vertices of the graph which are not dominated, in order to dominate v_{t-1}, v_t, v_{t+1} and v_t^+ vertices, v_t vertices should be taken into V_2, and v_{t-1}^+ and v_{t+1}^+ vertices should be taken into V_1 which satisfy $t \equiv 1(\text{mod } 3)$. Then $V_2 = \{v_1, v_4, v_7, \ldots, v_{n-4}, v_{n-2}\}$ and $V_1 = \{v_0^+, v_2^+, v_3^+, v_5^+, \ldots, v_{n-3}^+, v_{n-1}^+\}$.

Therefore $f(V) = (\frac{n-5-0}{3} + 1 + \frac{n-3-2}{3} + 1 + 1) + 2(\frac{n-4-1}{3} + 1 + 1)$ then we get $\gamma_R(P_n^+) \leq 4\lceil \frac{n}{3} \rceil - 1$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_1 = \{v \in V : f(v) = 1\}$, such as v_0^+ vertex. Since $f_1(v) \neq 2$ for $\forall v \in N(v_0^+)$, obtained function $f_1 = (V_0 \cup \{v_0^+\}, V_1 - \{v_0^+\}, V_2)$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$. For f_1 function to be an RDF, v_0^+ vertex should be taken into V_2; $f_1 = (V_0, V_1 - \{v_0^+\}, V_2 \cup \{v_0^+\})$. Hence we get $f_1(V) = 4\lceil \frac{n}{3} \rceil$. Since $f(V) < f_1(V)$ that $f_1(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$.

Let f not be a $\gamma_R - function$ and delete any vertex from $V_2 = \{v \in V : f(v) = 2\}$, such as v_1 vertex. Since $f_2(v) \neq 2$ for $\forall v \in N(v_1)$, obtained function $f_2 = (V_0 \cup \{v_1\}, V_1, V_2 - \{v_1\})$ does not satisfy the condition to be an RDF. According to this $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$. For f_2 function to be an RDF, v_0, v_1, v_2 and v_1^+ vertices should be taken into V_1; $f_2 = (V_0 - \{v_0, v_2, v_1^+\}, V_1 \cup \{v_0, v_1, v_2, v_1^+\}, V_2 - \{v_1\})$. Hence we get $f_2(V) = 4\lceil \frac{n}{3} \rceil + 1$. Since $f(V) < f_2(V)$ that $f_2(V) \neq \gamma_R(G)$. In this case $\gamma_R(P_n^+) \geq 4\lceil \frac{n}{3} \rceil - 1$. Consequently, we had $\gamma_R(P_n^+) = 4\lceil \frac{n}{3} \rceil - 1$.

Finally, we have obtained;

$$\gamma_R(P_n^+) = \begin{cases} 4\frac{n}{3} & n \equiv 0(\text{mod } 3) \\ 4\lfloor \frac{n}{3} \rfloor + 2 & n \equiv 1(\text{mod } 3) \\ 4\lceil \frac{n}{3} \rceil - 1 & n \equiv 2(\text{mod } 3) \end{cases}$$

□

3 Conclusion

The concept of Roman domination in graphs relates to dominating sets and the degree of vertices. The Roman domination numbers of some graphs already exist in the literature, where we have investigated the Roman domination numbers of the comet, double comet, and comb graphs. These graphs are important examples of tree-structured networks. Obtaining similar results for the other graph classes is an open area of research. Also, one may consider working on these graph structures for Italian domination, which could be considered as a variation of Roman domination.

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